A new approach in theoretical physics based on the Einstein covariance principle

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Abstract

In this article we show that Einstein covariance principle provides a wide opportunity in the solutions of different problems of theoretical physics. Here we apply covariance principle in some problems of classical electrodynamics and kinetics. Extension of this approach in the other fields is obvious.

1 INTRODUCTION

The requirement of general covariance of the equations describing different processes in the nature is one of the corner-stones of Einstein general relativity[1] and has enormous significance in modern theoretical physics. With the equivalence principle it brings gravitation to the metric properties of the space-time and shows connection of the geometry of space -time continuum with the material processes.

In this article we are going to open another powerful aspect of the Einstein covariance principle. Namely, we are going to show that covariance principle provides new methods for the solutions of different problems of theoretical physics.

Here we will apply principle of covariance to some problems of classical electrodynamics and kinetics. Extension of this approach to the other fields of physics is obvious.

The prehistory of this work is very short. More than twenty years ago one of the authors found out importance of Euler transformation in the classical electrodynamics[2]. Then, with use of the covariance principle, the boundary-value problem of the electrodynamics has been solved for an expanding-contracting sphere[3].

The paper is organized as follows. In Section 2 the Euler transformation is introduced, notations are given and preliminary explanations of the mathematical procedure in the article. In Section 3, with the help of covariance principle, we construct infinitely many solutions of the continuity equation from the given one. In Section 4 it is shown that the Euler transformation provides an opportunity to express charge and current densities of the ensemble of point particles in any arbitrary field as a linear combination of the same quantities in the absence of the external field. Application of the covariance principle in the Boltzman kinetic equation is given in Section 5. It is shown here that the Euler

transformation enables one to find out a solution of the nonrelativistic Boltzman equation in any external field if a solution of the same equation in the absence of the external field is known. Sections 6,7 are concerned with the problems of Maxwell's phenomenological electrodynamics. As an illustration of the developed technique we give a solution of the problem of propagation of electromagnetic waves in an inhomogeneous anisotropic medium. In Section 8 we consider a self- consistent problem of plasma electrodynamics. This Section shows the way of construction an exact solution of the interaction of electron plasma with the external electromagnetic field on the slab of positive ions.

2 BACKGROUND

a. In calculations we will use notations of the well known, Landau-Lifshitz book[4].

b. Throughout the article we will follow the same procedure, i.e.,we will consider an equation having some solution in an inertial reference frame K' with Cartesian coordinates $X'^{i}\left(ct',\mathbf{r}'\right)$ and metric tensor

$$g'^{ij} = g'_{ij} = \begin{pmatrix} 1000 \\ 0-100 \\ 00-10 \\ 000-1 \end{pmatrix}. \tag{1}$$

Then we perform an arbitrary transformation

$$X'^{i} = W^{i}\left(X\right), \quad i = 0,1,2,3. \tag{2}$$

to a noninertial reference frame K with the covariant form of given equation. Just comparing the forms of the equation in K' and K we find a new set of solutions of the given equation in the inertial frame K'.

c. In the noninertial frame K covariant components of metric tensor are

$$g_{ij} = \Lambda_i^m \Lambda_j^n g'_{mn}, \tag{3}$$

where

$$\Lambda_j^i\left(X\right) = \frac{\partial X'^{i}}{\partial X^{j}} \equiv \partial_j W^{i}\left(X\right) \tag{4}$$

is the matrix of transformation (2) .

The contravariant components of metric tensor of K is given by expression

$$g^{ij} = \tilde{\Lambda}_m^i \tilde{\Lambda}_n^j g'^{mn}, \tag{5}$$

where $\tilde{\Lambda}$ is the reciprocal matrix of Λ

$$\Lambda_m^i \tilde{\Lambda}_j^m = \tilde{\Lambda}_m^i \Lambda_j^m = \delta_j^i. \tag{6}$$

d. We use Euler transformation which is very well known in the theory of elasticity as

$$\mathbf{r}'= \mathbf{r} - \mathbf{u}\,(\mathbf{r},t) \tag{7}$$

where $\mathbf{u}\,(\mathbf{r},t)$ is the displacement field of medium. Here we will consider four-dimensional form of (7):

$$\begin{aligned} X'^i &= X^i - U^i\,(X)\,, \\ U^0 &= 0. \end{aligned} \tag{8}$$

e. With the help of (4) and (8) we have the following expression of Λ, $\tilde{\Lambda}$ for Euler transformation:

$$\Lambda^0_0 = 1,\ \Lambda^0_\alpha = 0,\ \Lambda^\alpha_0 = -\frac{\dot{u}_\alpha}{c},\ \Lambda^\alpha_\beta = \delta_{\alpha\beta} - \partial_\beta u_\alpha \equiv S_{\alpha\beta}, \tag{9}$$

$$\tilde{\Lambda}^0_0 = 1,\ \tilde{\Lambda}^0_\alpha = 0,\ \tilde{\Lambda}^\alpha_0 = \frac{\dot{u}_\beta}{c} S^{-1}_{\alpha\beta},\ \tilde{\Lambda}^\alpha_\beta = S^{-1}_{\alpha\beta}. \tag{10}$$

Here u_α $(\alpha = 1, 2, 3)$ are components of $\mathbf{u}$, $S^{-1}_{\alpha\beta}$ is reciprocal matrix of $S_{\alpha\beta}$, and

$$\frac{\dot{u}_\alpha}{c} \equiv \partial_0 u_\alpha.$$

Using (3) , (5) and (9), (10) we have for the covariant and contravariant components of the metric tensor in reference frame K, respectively

$$g_{00} = 1 - \frac{\dot{\mathbf{u}}^2}{c^2},\ g_{0\alpha} = \frac{\dot{u}_\beta}{c} S_{\beta\alpha},\ g_{\alpha\beta} = -S_{\lambda\alpha} S_{\lambda\beta}, \tag{11}$$

$$g^{00} = 1,\ g^{0\alpha} = \frac{\dot{u}_\beta}{c} S^{-1}_{\alpha\beta},\ g^{\alpha\beta} = -S^{-1}_{\alpha\sigma} S^{-1}_{\beta\lambda} (\delta_{\sigma\lambda} - \frac{\dot{u}_\sigma \dot{u}_\lambda}{c^2}). \tag{12}$$

Then, for the determinant of covariant metric tensor we get:

$$\sqrt{-g} = 1 - \partial_\lambda a_\lambda, \tag{13}$$

$$a_\lambda = u_\lambda + \frac{1}{2}\left[u_\nu \partial_\nu u_\lambda - u_\lambda \partial_\nu u_\nu\right] + \frac{1}{3}\sigma_{\lambda\nu} u_\nu, \tag{14}$$

$$\sigma_{\alpha\beta} = \frac{1}{2} e_{\alpha\mu\nu} e_{\beta\rho\lambda} (\partial_\mu u_\rho)(\partial_\nu u_\lambda). \tag{15}$$

Space components of contravariant metric tensor are determined in (12) via expressions:

$$\sqrt{-g} S^{-1}_{\alpha\beta} = \delta_{\alpha\beta} - \partial_\nu b_{\nu\alpha\beta}, \tag{16}$$

$$b_{\nu\alpha\beta} = u_\nu \delta_{\alpha\beta} - u_\alpha \delta_{\nu\beta} - \frac{1}{2} u_\lambda e_{\lambda\beta\sigma} e_{\nu\alpha\mu} \partial_\mu u_\sigma. \tag{17}$$

Here $e_{\alpha\beta\gamma}$, $\delta_{\alpha\beta}$ are three dimensional Levi-Civita and Kronecker symbols respectively.

Finally, with the help of (11), (12) we get the following expressions for the nonzero components of Christoffel symbols for Euler transformation (8)

$$\Gamma^\alpha_{00} = -\frac{\ddot{u}_\nu}{c^2} S^{-1}_{\alpha\nu},\ \Gamma^\alpha_{0\beta} = -\frac{(\partial_\beta \dot{u}_\nu)}{c} S^{-1}_{\alpha\nu},\ \Gamma^\alpha_{\lambda\sigma} = -(\partial_\lambda \partial_\sigma u_\nu) S^{-1}_{\alpha\nu}\ . \tag{18}$$

3 CONTINUITY EQUATION

For simplicity let us start from continuity equation, which in an inertial reference frame K' has the form

$$\partial'_i j'^i = 0. \tag{19}$$

After transformation (2) to noninertial reference frame K we have covariant continuity equation:

$$\frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g} j^i \right) = 0. \tag{20}$$

Here $\sqrt{-g}$ is the determinant of the transformation matrix (4). On the other hand, because of the four-vector character of the current, we have

$$j^i \left(X \right) = \tilde{\Lambda}_n^i \left(X \right) j'^n \left(W \left(X \right) \right). \tag{21}$$

By comparing (19) with (20) and taking into account (21), we may state the following. If $j_0^i \left(X \right)$ is a solution of the continuity equation

$$\partial_i j_0^i \left(X \right) = 0, \tag{22}$$

then

$$j_1^i \left(X \right) \equiv \sqrt{-g \left(X \right)} \tilde{\Lambda}_n^i \left(X \right) j_0^n \left(W \left(X \right) \right) \tag{23}$$

satisfies as well the same equation (22) for any functions $W^i \left(X \right)$.

Thus, the covariance principle provides an opportunity to construct infinitely many solutions (23) of continuity equation (22) from a given one.

4 PARTICLE CHARGE AND CURRENT DENSITIES IN THE EXTERNAL FIELD

In this Section we show that expression (23) has very important consequences in the electrodynamics. It provides an opportunity to express charge and current densities in the arbitrary external field via charge and current densities of the undisturbed system.

Let us consider an ensemble of identical particles in a medium, having charge e and trajectories $\mathbf{r}_a^0 \left(t \right)$ $\left(a = 1, 2, ... \right)$ in the absence of external field. Then the current four-vector at the point $X'^i \equiv \left(ct', \mathbf{r}' \right)$ is given by[4]

$$j_0^i \left(X' \right) = ce \sum_a \delta \left(\mathbf{r}' - \mathbf{r}_a^0 \left(t' \right) \right) \frac{dX'^i}{dX'^0}. \tag{24}$$

In the presence of an external field the trajectories of particles are changed $\mathbf{r}_a^0(t) \rightarrow \mathbf{r}_a(t)$ and the current at the point X is

$$j^i(X) = ce \sum_a \delta(\mathbf{r} - \mathbf{r}_a(t)) \frac{dX^i}{dX^0}. \tag{25}$$

Now, if we perform the Euler transformation in (24) and use the well-known formula

$$\prod_{i=1}^{n} \delta(x_i - \alpha_i) = \frac{1}{|J|} \prod_{i=1}^{n} \delta(\xi_i - \beta_i), \; J \equiv \frac{\partial(x_1....x_n)}{\partial(\xi_1.....\xi_n)}$$

we arrive at the expression

$$j^i(X) \equiv \sqrt{-g(X)} \tilde{\Lambda}_n^i(X) j_0^n(X - U), \tag{26}$$

under the condition

$$\mathbf{u}(\mathbf{r}_a(t), t) = \mathbf{r}_a(t) - \mathbf{r}_a^0(t). \tag{27}$$

This means that in an arbitrary external field the current four-vector $j^i(X)$ is expressed linearly in terms of the undisturbed four-current j_0^i at the point $X - U$.

As we see, $\mathbf{u}(\mathbf{r},t)$ is a field which on the trajectories of particles equals to the displacement of trajectories of particles caused by external forces and, hence, it has the analogous meaning as in the theory of elasticity.

The expression (26) is a special case of (23) when $W^i(X)$ is the Euler transformation with the additional condition (27). Hence, we can say that, by using the covariance principle, we are able to connect current and charge densities of disturbed and undisturbed system of identical point particles in any external field with the help of the Euler transformation.

In three-dimensional form Eq.(26) is presented as[2]

$$\rho(\mathbf{r}, t) = \sqrt{-g} \rho_0(\mathbf{r} - \mathbf{u}, t), \tag{28}$$

$$j_\alpha(\mathbf{r}, t) = \sqrt{-g} S_{\alpha\beta}^{-1} [\dot{u}_\beta \rho_0(\mathbf{r} - \mathbf{u}, t) + j_{0\beta}(\mathbf{r} - \mathbf{u}, t)], \tag{29}$$

where we used (10).Here $\sqrt{-g}$ and $S_{\alpha\beta}^{-1}$ are given by (13), (16).

In the applications of theory we have often situations where the charges are initially uniformly distributed and there is no current, i.e.,

$$\begin{aligned} \rho_0 &= const, \\ \mathbf{j}_0 &= 0. \end{aligned} \tag{30}$$

In this case, with the help of (13), (16), we can present (28), (29) in the familiar way

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_0 - \nabla \mathbf{P}(\mathbf{r},t), \\ \mathbf{j}(\mathbf{r},t) &= \frac{\partial}{\partial t} \mathbf{P}(\mathbf{r},\mathbf{t}) + [\nabla \times \mathbf{M}(\mathbf{r},t)]. \end{aligned} \tag{31}$$

Here the electric and magnetic polarization vectors have the forms

$$P_\alpha = \rho_0 \{ u_\alpha + \frac{1}{2} [u_\nu \partial_\nu u_\alpha - u_\alpha \partial_\nu u_\nu] + \frac{1}{3} \sigma_{\alpha\nu} u_\nu \}, \tag{32}$$

$$M_\alpha = \rho_0 \{ \frac{1}{2} e_{\alpha\lambda\nu} u_\lambda \dot{u}_\nu + \frac{1}{3} e_{\nu\sigma\lambda} \dot{u}_\nu u_\sigma \partial_\alpha u_\lambda \}. \tag{33}$$

Hence, in this case we can introduce electric and magnetic polarizations in an unambiguous, natural way.

5 BOLTZMAN EQUATION

Suppose that in an inertial reference system K' the distribution function $f'(X', P')$ of the particles with charge e and mass m satisfies the relativistic Boltzman equation[5]

$$P'^i \partial'_i f' \left(X', P' \right) = C' \left(X', P' \right) \tag{34}$$

where $C'(X', P')$ is the collision integral and P'^i is the momentum four-vector of particles. In noninertial reference system K we have covariant Boltzman equation[6]

$$\left[P^i \partial_i + m F^i (X, P) \frac{\partial}{\partial P^i} \right] f (X, P) = C (X, P), \tag{35}$$

where $F^i(X, P)$ is the inertial force given by

$$F^i (X, P) = -\frac{1}{m} \Gamma^i_{jl} P^j P^l. \tag{36}$$

Distribution function and collision integrals are scalars i.e.

$$f' \left(X', P' \right) = f (X, P), \tag{37}$$

$$C' \left(X', P' \right) = C (X, P), \tag{38}$$

where

$$X'^i = W^i (X), \tag{39}$$

$$P^i = \tilde{\Lambda}^i_n P'^n. \tag{40}$$

Then, from (34) -(40), one can say that if $f_0(X, P)$ is a solution of the free Boltzman equation

$$P^i \partial_i f_0 (X, P) = C_0 (X, P), \tag{41}$$

then

$$\bar{f} (X, P) = f_0 \left(W (X), \check{\Lambda} (X) P \right) \tag{42}$$

satisfies the Boltzman equation in the external field $F^i(X,P)$ (36) with collision integral

$$\bar{C}(X,P) = C_0\left(W(X), \check{\Lambda}(X)P\right). \tag{43}$$

In the last two expressions (42) ,(43) we have introduced short-hand writing $\check{\Lambda}(X)P$ instead of $\Lambda^i_m(X)P^m$.

Taking into account difference of the values of phase volumes in K' and K we obtain that the distribution function corresponding to the solution (42) is

$$f(X,P) = \left(\sqrt{-g(X)}\right)^2 f_0\left(W(X), \check{\Lambda}(X)P\right). \tag{44}$$

As the four-current of particles is defined as

$$j^i(X) = \frac{e}{m}\int dPP^i f(X,P) \tag{45}$$

with distribution function (44), we get, by changing integration variables $P^i \to \Lambda^i_m P^m$, the same expression as (23) with

$$j^i_0(X) \equiv \frac{e}{m}\int dPP^i f_0(X,P) \tag{46}$$

as it must be.

Now let us consider these results for the Euler transformation in nonrelativistic limit, i.e., for

$$P^i = (mc, \mathbf{p}), \tag{47}$$

where $\mathbf{p} \equiv m\mathbf{v}$ and $\mathbf{v}$ is the particle velocity satisfying Newton's equation of motion:

$$m\dot{\mathbf{v}}(t) = \mathbf{F}. \tag{48}$$

Insertion of (47) into (36) and use of expressions for Christoffel symbols (18) for Euler transformation gives the following expression for equation (35)

$$\{\frac{\partial}{\partial t} + (\mathbf{v}\nabla) + mS^{-1}_{\alpha\nu}\left[\ddot{u}_\nu + 2(\mathbf{v}\nabla)\dot{u}_\nu + (\mathbf{v}\nabla)^2 u_\nu\right]\frac{\partial}{\partial p_\alpha}\}\bar{f}(\mathbf{r},\mathbf{p},t) = C(\mathbf{r},\mathbf{p},t), \tag{49}$$

where

$$\bar{f}(\mathbf{r},\mathbf{p},t) \equiv f_0\left(X-U, \check{\Lambda}P\right) \tag{50}$$

$$C(\mathbf{r},\mathbf{p},t) = \frac{1}{m}C_0\left(X-U, \check{\Lambda}P\right) \tag{51}$$

On the other hand,

$$\ddot{u}_\nu + 2(\mathbf{v}\nabla)\dot{u}_\nu + (\mathbf{v}\nabla)^2 u_\nu + (\dot{\mathbf{v}}\nabla)u_\nu \equiv \frac{d^2}{dt^2}u_\nu(\mathbf{r}(t),t) \tag{52}$$

so, taking into account (48), we have from (49)

$$\{\frac{\partial}{\partial t}+(\mathbf{v}\nabla)+\mathbf{F}\frac{\partial}{\partial \mathbf{p}}\}\bar{f}\left(\mathbf{r},\mathbf{p},t\right)=C\left(\mathbf{r},\mathbf{p},t\right) \tag{53}$$

provided the condition

$$m\frac{d^2}{dt^2}\mathbf{u}\left(\mathbf{r}\left(t\right),t\right)=\mathbf{F} \tag{54}$$

is met.

This condition is equivalent to (27) and may be written as

$$\mathbf{u}\left(\mathbf{r}\left(t\right),t\right)=\mathbf{r}\left(t\right)-\mathbf{r}^0\left(t\right), \tag{55}$$

where $\mathbf{r}^0\left(t\right)$ is undisturbed trajectory of the particle (i.e. switching of the external field $\mathbf{F}$ replaces $\mathbf{r}^0\left(t\right)$ by $\mathbf{r}\left(t\right)$).

The equation (53) is seen to be the Boltzman equation in the external field of the force $\mathbf{F}$ and so we can confirm once more that having any solution of free Boltzman equation (41) we are able to construct a new solution (50) for the same equation in an arbitrary external field (53) in nonrelativistic limit.

Let us now suppose that the external field is switched on at $t=0$. In this case the problem above corresponds to the Cauchy problem for Boltzman equation and it may be stated that we have solved the Cauchy problem for Boltzman equation in an external field in nonrelativistic limit.

Finally, we would like to recall that the real distribution function corresponding to the solution (50) is given by

$$f\left(\mathbf{r},\mathbf{p},t\right)=\left(\sqrt{-g}\right)^2 f_0\left(\mathbf{r}-\mathbf{u},\check{\Lambda}P,t\right) \tag{56}$$

as mentioned above.

6 OPEN ELECTRODYNAMICS

Let us start from phenomenological electrodynamics in an inertial frame K' with Cartesian coordinates, i.e., from Maxwell equations in Minkowski representation

$$\partial'_j H'^{ij}=-\frac{4\pi}{c}j'^{i}_{ext}, \tag{57}$$

$$\partial'_j F'_{il}+\partial'_i F'_{lj}+\partial'_l F'_{ji}=0, \tag{58}$$

and the most general phenomenological expansion

$$H'^{ij}\left(X'\right)=\sum_{s=1}^{\infty}\int dX'_1..dX'_s\varepsilon^{ijl_1m_1...l_sm_s}\left(X',X'_1,..X'_s\right)F'_{l_1m_1}\left(X'_1\right)...F'_{l_sm_s}\left(X'_s\right). \tag{59}$$

Here $H^{ij}=(-\mathbf{D},\mathbf{H})$, $F_{ij}=(\mathbf{E},\mathbf{B})$ are Minkowski tensors of electromagnetic field, j^i_{ext} is the four-current of external charges, $\varepsilon^{ijl_1m_1...l_sm_s}(X,X_1,..X_s)$ is a $2(s+1)$-rank tensor describing electromagnetic properties of the medium (tensor of electromagnetic permittivity).

In the frame K, after transformation (2) , Maxwell equations (57), (58) are

$$\frac{1}{\sqrt{-g}}\partial_j(\sqrt{-g}H^{ij})=-\frac{4\pi}{c}j^i_{ext}, \tag{60}$$

$$\partial_jF_{il}+\partial_iF_{lj}+\partial_lF_{ji}=0, \tag{61}$$

$$\begin{aligned} H^{ij}(X) &= \sum_{s=1}^{\infty}\int dX_1..dX_s\varepsilon^{ijl_1m_1...l_sm_s}(X,X_1,..X_s)\times & (62)\\ &\times\sqrt{-g(X_1)}F_{l_1m_1}(X_1)...\sqrt{-g(X_s)}F_{l_sm_s}(X_s), & (63)\end{aligned}$$

where

$$\begin{aligned}\varepsilon^{ijl_1m_1...l_sm_s}(X,X_1,..X_s) &= \tilde{\Lambda}^i_{i'}(X)\tilde{\Lambda}^j_{j'}(X)\tilde{\Lambda}^{l_1}_{l_1'}(X_1)\tilde{\Lambda}^{m_1}_{m_1'}(X_1)...\tilde{\Lambda}^{l_s}_{l_s'}(X_s)\tilde{\Lambda}^{m_s}_{m_s'}(X_s)\times\\ &\times\varepsilon^{i'j'l_1'm_1'...l_s'm_s'}(W(X),W(X_1),..W(X_s)). \end{aligned}\tag{64}$$

By defining

$$\bar{H}^{ij}(X)=\sqrt{-g(X)}\tilde{\Lambda}^i_m(X)\tilde{\Lambda}^j_n(X)H^{mn}(W(X)), \tag{65}$$

$$\bar{F}_{ij}(X)=\Lambda^m_i(X)\Lambda^n_j(X)F_{mn}(W(X)), \tag{66}$$

$$\bar{j}^i_{ext}(X)=\sqrt{-g(X)}\tilde{\Lambda}^i_m(X)j^m_{ext}(W(X)), \tag{67}$$

we can state, after comparing (57), (58) with (60), (61): if $\check{H}$, $\check{F}$ tensors satisfy Maxwell equations

$$\partial_jH^{ij}=-\frac{4\pi}{c}j^i_{ext},\quad \partial_jF_{il}+\partial_iF_{lj}+\partial_lF_{ji}=0, \tag{68}$$

with material equation

$$H^{ij}(X)=\sum_{s=1}^{\infty}\int dX_1..dX_s\varepsilon^{ijl_1m_1...l_sm_s}(X,X_1,..X_s)F_{l_1m_1}(X_1)...F_{l_sm_s}(X_s), \tag{69}$$

then (65)-(67) satisfy the same set of equations with material equation

$$\bar{H}^{ij}(X)=\sum_{s=1}^{\infty}\int dX_1..dX_s\bar{\varepsilon}^{ijl_1m_1...l_sm_s}(X,X_1,..X_s)\bar{F}_{l_1m_1}(X_1)...\bar{F}_{l_sm_s}(X_s). \tag{70}$$

and the electromagnetic permittivity tensor

$$\bar{\varepsilon}^{ijl_1m_1....l_sm_s}(X,X_1....X_s) \equiv \gamma^{ij}_{i'j'}(X)\gamma^{l_1m_1}_{l'_1m'_1}(X_1)...\gamma^{l_sm_s}_{l'_sm'_s}(X_s)\times$$
$$\times\varepsilon^{i'j'l'_1m'_1...l'_sm'_s}(W(X),W(X_1),..W(X_s)). \quad (71)$$

Here

$$\gamma^{ij}_{mn}(X) \equiv \sqrt{-g(X)}\tilde{\Lambda}^i_m(X)\tilde{\Lambda}^j_n(X). \quad (72)$$

So, having a solution of Maxwell equations in a medium with a certain permittivity tensor $\varepsilon^{ijl_1m_1...l_sm_s}(X,X_1,..X_s)$ we are able to construct infinitely many other exact solutions of the Maxwell equations in the media with permittivity tensors $\bar{\varepsilon}^{ijl_1m_1....l_sm_s}(X,X_1....X_s)$.

In the case where the initial phenomenological equation (69) is local one, i.e., if

$$H^{ij}(X) = \sum_{s=1}^{\infty}\varepsilon^{ijl_1m_1...l_sm_s}(X)F_{l_1m_1}(X)...F_{l_sm_s}(X), \quad (73)$$

then it is very easy to show that (65)-(67) satisfy the Maxwell equations with material equation

$$\bar{H}^{ij}(X) = \sum_{s=1}^{\infty}\bar{\varepsilon}^{ijl_1m_1...l_sm_s}(X)\bar{F}_{l_1m_1}(X)...\bar{F}_{l_sm_s}(X), \quad (74)$$

and the electromagnetic permittivity tensor:

$$\bar{\varepsilon}^{ijl_1m_1...l_sm_s}(X) = \sqrt{-g(X)}\tilde{\Lambda}^i_{i'}(X)\tilde{\Lambda}^j_{j'}(X)\tilde{\Lambda}^{l_1}_{l'_1}(X)\tilde{\Lambda}^{m_1}_{m'_1}(X)...\tilde{\Lambda}^{l_s}_{l'_s}(X)\tilde{\Lambda}^{m_s}_{m'_s}(X)\times$$
$$\times\varepsilon^{i'j'l'_1m'_1...l'_sm'_s}(W(X)). \quad (75)$$

7 LOCAL LINEAR PROBLEMS

In this case (73) has the following appearance:

$$H^{ij}(X) = \varepsilon^{ijmn}(X)F_{mn}(X), \quad (76)$$

and for (75) we have

$$\bar{\varepsilon}^{ijlm}(X) = \sqrt{-g(X)}\tilde{\Lambda}^i_{i'}(X)\tilde{\Lambda}^j_{j'}(X)\tilde{\Lambda}^l_{l'}(X)\tilde{\Lambda}^m_{m'}(X)\varepsilon^{i'j'l'm'}(W(X)). \quad (77)$$

In three-dimensional notations (76) becomes

$$\begin{aligned} D_\alpha &= -2\varepsilon^{0\alpha0\beta}E_\beta + \varepsilon^{0\alpha\beta\gamma}e_{\beta\gamma\sigma}B_\sigma, \\ H_\alpha &= \frac{1}{2}e_{\alpha\beta\gamma}\varepsilon^{\beta\gamma\lambda\mu}e_{\lambda\mu\sigma}B_\sigma - e_{\alpha\beta\gamma}\varepsilon^{\beta\gamma0\sigma}E_\sigma. \end{aligned} \quad (78)$$

For ordinary media

$$\begin{aligned} D_\alpha &= \varepsilon_{\alpha\beta}E_\beta, \\ B_\alpha &= \mu_{\alpha\beta}H_\beta. \end{aligned} \quad (79)$$

Then (78), (79) give for nonzero components of $\varepsilon^{\beta\gamma\lambda\mu}$

$$\begin{aligned} \varepsilon^{0\alpha\beta 0} &= \frac{1}{2}\varepsilon_{\alpha\beta}, \\ \varepsilon^{\alpha\beta\lambda\mu} &= \frac{1}{2}e_{\alpha\beta\nu}\mu^{-1}_{\nu\sigma}e_{\sigma\lambda\mu}. \end{aligned} \tag{80}$$

Let's consider time independent transformation (2), i.e.,

$$\mathbf{r}' = \mathbf{W}(\mathbf{r}), \, t' = t; \tag{81}$$

then for nonzero components of transformation matrices Λ, $\tilde{\Lambda}$ we have from (4)

$$\Lambda^0_0 = \tilde{\Lambda}^0_0 = 1, \; \Lambda^\alpha_\beta = \partial_\beta W_\alpha \equiv M_{\alpha\beta}, \; \tilde{\Lambda}^\alpha_\beta = M^{-1}_{\alpha\beta}, \; \sqrt{-g} = \|M\| \tag{82}$$

and, hence, for the new dielectric and magnetic tensors we obtain from (77), (80) and (82)

$$\bar{\varepsilon}_{\alpha\beta}(\mathbf{r}) = \|M\| \, M^{-1}_{\alpha\nu}M^{-1}_{\beta\lambda}\varepsilon_{\nu\lambda}(\mathbf{W}(\mathbf{r})), \tag{83}$$

$$\bar{\mu}_{\alpha\beta}(\mathbf{r}) = \|M\| \, M^{-1}_{\alpha\nu}M^{-1}_{\beta\lambda}\mu_{\nu\lambda}(\mathbf{W}(\mathbf{r})). \tag{84}$$

From (65)-(67) , (82) we find the solutions of Maxwell equations in a medium with $\bar{\varepsilon}_{\alpha\beta}(\mathbf{r}), \bar{\mu}_{\alpha\beta}(\mathbf{r})$ as:

$$\bar{E}_\alpha(\mathbf{r},t) = M_{\beta\alpha}E_\beta(\mathbf{W}(\mathbf{r}), t) \tag{85}$$

$$\bar{B}_\alpha(\mathbf{r},t) = \|M\| \, M^{-1}_{\alpha\beta}B_\beta(\mathbf{W}(\mathbf{r}), t) \tag{86}$$

in the presence of the following external sources:

$$\bar{\rho}_{ext}(\mathbf{r},t) = \|M\| \, \rho_{ext}(\mathbf{W}(\mathbf{r}), t), \tag{87}$$

$$[\bar{j}_{ext}(\mathbf{r},t)]_\alpha = \|M\| \, M^{-1}_{\alpha\beta}[j_{ext}(\mathbf{W}(\mathbf{r}), t)]_\beta. \tag{88}$$

As an illustration of our results let us construct a new solution of Maxwell equations in a medium if we have these solutions in vacuum ($j_{ext} \equiv 0$).

For the vacuum

$$\varepsilon_{\alpha\beta} = \mu_{\alpha\beta} = \delta_{\alpha\beta} \tag{89}$$

and, with the help of transformation (81) and (83), (84) ,(89) , we come to a medium with equal dielectric and magnetic permittivity tensors:

$$\bar{\varepsilon}_{\alpha\beta}(\mathbf{r}) = \bar{\mu}_{\alpha\beta}(\mathbf{r}) = \|M\| \, M^{-1}_{\alpha\nu}M^{-1}_{\beta\nu}. \tag{90}$$

As the simplest case of transformation (81) let us take

$$x' = x, \quad y' = y, \quad z' = f(z), \quad t' = t \tag{91}$$

where $f(z)$ is an arbitrary monotonically increasing function

$$f'(z) \equiv n(z) > 0. \tag{92}$$

From (82) we have for nonzero matrix elements of $\hat{M}$

$$M_{xx} = M_{yy} = 1, \quad M_{zz} = n(z),$$

and (90) gives

$$\bar{\varepsilon}_{xx} = \bar{\varepsilon}_{yy} = \bar{\mu}_{xx} = \bar{\mu}_{yy} = n(z), \quad \bar{\varepsilon}_{zz} = \bar{\mu}_{zz} = \frac{1}{n(z)} \tag{93}$$

From (85),(86) we have for the solutions of Maxwell equations in the medium with equal electrical and magnetic tonsorial permittivity (93)

$$\begin{aligned} \bar{E}_\alpha(\mathbf{r},t) &= E_\alpha(x,y,f(z),t), \quad \alpha = x,y \\ \bar{E}_z(\mathbf{r},t) &= n(z) E_z(x,y,f(z),t) \end{aligned} \tag{94}$$

$$\begin{aligned} \bar{B}_\alpha(\mathbf{r},t) &= n(z) B_\alpha(x,y,f(z),t), \quad \alpha = x,y \\ \bar{B}_z(\mathbf{r},t) &= B_z(x,y,f(z),t), \end{aligned} \tag{95}$$

where $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$ are any solutions of Maxwell equation in vacuum (for instance, harmonic functions).

Up to now we knew solution of Maxwell equations for media with refractive index with constant, linear, quadratic and exponential coordinate dependence. Our approach extends these solutions to anisotropic media with primitivities containing arbitrary function (93).

8 ELECTRON PLASMA ON THE BACKGROUND OF IONS IN A SLAB

In this section we intend to show how to use the developed technique in construction of a solution for the self-consistent problem of plasma interacting with electromagnetic field (EMF).

Consider two component plasma (electrons and ions). For simplicity suppose that ions are very heavy and interaction with EMF does not change their state of equilibrium. Besides, let us also assume that by some reason they cannot leave the borders of a slab $[-a \leq x \leq a]$. Then, we come to the problem of electron plasma on the positively charged background in a slab.

The charge density in the slab of positive background is:

$$\rho_0(x) = -en[\theta(x+a) - \theta(x-a)], \tag{96}$$

where n is the concentration of ions. Then, in the presence of external EMF for electron plasma we have Maxwell-Boltzman equations:

$$\partial_j F^{ij} = -\frac{4\pi}{c}(j^i + j_0^i), \tag{97}$$

$$\partial_j F_{il} + \partial_i F_{lj} + \partial_l F_{ji} = 0, \tag{98}$$

$$\left[P^i \partial_i + \frac{e}{mc} F^{ij}\left(X\right) P_j \frac{\partial}{\partial P^i} \right] f\left(X, P\right) = C\left(X, P\right). \tag{99}$$

Here the electron current four-vector is defined as

$$j^i\left(X\right) = \frac{e}{m} \int dP P^i f\left(X, P\right), \tag{100}$$

while the ion current four-vector is

$$j_0^i\left(x\right) \equiv \left(c\rho_0\left(x\right), 0, 0, 0\right). \tag{101}$$

Let us search for the solution of (97)-(99) as:

$$F^{ij} = F^{ij}_{ion} + F^{ij}_e, \tag{102}$$

where F^{ij}_{ion} is the static field of ions having the charge distribution (96) in the form

$$F^{10}_{ion} = -F_{ion10} = -4\pi en\left[x\theta\left(a - |x|\right) + a\theta\left(|x| - a\right)\right]. \tag{103}$$

Substituting (102) into (97)-(99) results in

$$\partial_j F^{ij}_e = -\frac{4\pi}{c} j^i \tag{104}$$

$$\partial_j F_{eil} + \partial_i F_{elj} + \partial_l F_{eji} = 0, \tag{105}$$

$$\left[P^i \partial_i + \frac{e}{mc}(F^{ij}_e + F^{ij}_{ion}) P_j \frac{\partial}{\partial P^i} \right] f\left(X, P\right) = C\left(X, P\right). \tag{106}$$

Now our problem is to construct the solution for (104)-(106).

Let us start from the equations:

$$\partial_j F^{(0)ij} = -\frac{4\pi}{c} j^{(0)i}, \tag{107}$$

$$P^i \partial_i f^{(0)} = C_0. \tag{108}$$

Here $f^{(0)}$ is the electron distribution function in equilibrium which satisfies the neutrality condition

$$j^{(0)i} = \frac{e}{m} \int dP P^i f^{(0)} \equiv -j_0^i\left(x\right). \tag{109}$$

Hence, as a solution of (107) we can take

$$F^{(0)ij} = -F^{ij}_{ion} + \bar{F}^{ij}, \tag{110}$$

where $\bar{F}^{ij}$ is a solution of Maxwell equations in vacuum.
After the Euler transformation in (107), (108) with $\mathbf{u}\left(\mathbf{r},t\right)$ satisfying the equation

$$\frac{d^2}{dt^2}u_\nu\left(\mathbf{r}\left(t\right),t\right) = \frac{e}{m^2c}F^{\nu j}\left(\mathbf{r}\left(t\right),t\right)P_j, \tag{111}$$

the equation (107) becomes

$$\partial_j(\sqrt{-g}F^{ij}) = -\frac{4\pi}{c}j^i, \tag{112}$$

where

$$F^{ij}\left(X\right) = \tilde{\Lambda}^i_m\tilde{\Lambda}^j_n F^{(0)mn}\left(X-U\right), \tag{113}$$

$$j^i\left(X\right) = \sqrt{-g}\tilde{\Lambda}^i_m j^{(0)m}\left(X-U\right). \tag{114}$$

Besides, after this transformation (108) goes to (106) with electron distribution function

$$f\left(X,P\right) = \left(\sqrt{-g}\right)^2 f^{(0)}\left(X-U,\check{\Lambda}P\right). \tag{115}$$

Because of (113) solution of (112) is

$$\tilde{F}^{ij}\left(X\right) = \sqrt{-g}\tilde{\Lambda}^i_m\tilde{\Lambda}^j_n F^{(0)mn}\left(X-U\right). \tag{116}$$

Then, with the help of the formula (125) of the Appendix and (102) ,(116) , we come to the following solution for EMF

$$F^{ij}\left(X\right) = F^{ij}_{ion}\left(X\right) + \tilde{F}^{ij}\left(X\right) + \frac{1}{2}e^{ijpq}e_{slmn}\partial_p\int dX_1 G^s_q\left(X,X_1\right)\partial^l_1\tilde{F}^{mn}\left(X_1\right). \tag{117}$$

So, the solutions of Maxwell-Boltzman equations, (97)-(99) , for electron plasma on the positive background in a slab in the external EMF is given by Eq.(115) for electron distribution function and by Eq.(117) for EMF tensor.

SUMMARY

With the help of the Einstein covariance principle we succeeded in:

a. constructing infinitely many solutions to the continuity equation, Boltzman equation and Maxwell phenomenological equations, if we have some single solution of these equations.

b. obtaining a general expression for the charge and current densities of the system of charged point particles in an arbitrary external field.

c. solving the Cauchy problem for the nonrelativistic Boltzman equation in an arbitrary external field.

d. solving the Maxwell equations in an anisotropic inhomogenious medium with equal electrical and magnetic primitivities.

e. demonstrating an algoritm for construction of a solution of the self-consistent problem of the interaction of electron plasma with the external electromagnetic field on the slab of positive ions.

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APPENDIX

Let $\tilde{F}^{ij}$ be an antisymmetric tensor satisfying the first Maxwell equation:

$$\partial_j \tilde{F}^{ij} = -\frac{4\pi}{c} j^i. \tag{118}$$

Problem is in construction of an antisymmetric tensor F^{ij} satisfying both Maxwell equations having $\tilde{F}^{ij}$.

Solution: Because $\tilde{F}^{ij}$ satisfy (118) , for an arbitrary four-vector A^i

$$F^{ij}(X) = \tilde{F}^{ij}(X) + e^{ijpq} \partial_p A_q(X) \tag{119}$$

is as well a solution of (118) .

Let us claim (119) to be a solution of the second Maxwell equation, i.e.,

$$e_{ijkl} \partial^j F^{kl}(X) = 0. \tag{120}$$

From (119) , (120) , we get equation for A_j:

$$\left[\partial_i \partial^j - \delta_i^j \Box_X\right] A_j(X) = \partial^j \tilde{F}^*_{ij}(X), \tag{121}$$

where $\tilde{F}^*_{ij}$ is dual tensor of $\tilde{F}_{ij}$

$$\tilde{F}^*_{ij} = \frac{1}{2} e_{ijmn} \tilde{F}^{mn}. \tag{122}$$

Let $G^l_j(X, X')$ be the Green's function of the wave equation

$$\left[\partial_i \partial^j - \delta_i^j \Box_X\right] G^l_j(X, X') = \delta^l_i \delta(X - X'). \tag{123}$$

Then, taking A_j as a solution of inhomogenious equation (121) ,

$$A_j(X) = \int dX_1 G^i_j(X, X_1) \partial^l_1 \tilde{F}^*_{il}(X_1) \tag{124}$$

we will satisfy the second Maxwell equation for the tensor F^{ij} given by (119) .

Hence, from (119) ,(121), (122) and (124) we finally get the expression for F^{ij} satisfying both Maxwell equations

$$F^{ij}\left(X\right)=\tilde{F}^{ij}\left(X\right)+\frac{1}{2}e^{ijpq}e_{slmn}\partial_{p}\int dX_{1}G_{q}^{s}\left(X,X_{1}\right)\partial_{1}^{l}\tilde{F}^{mn}\left(X_{1}\right). \tag{125}$$

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